Design of Engineering Experiments Part 4 – Introduction to Factorials

- Text reference, Chapter 5
- **General principles** of factorial experiments
- The **two-factor factorial** with fixed effects
- The **ANOVA** for factorials
- Extensions to more than two factors
- **Quantitative** and **qualitative** factors – response curves and surfaces

Some Basic Definitions

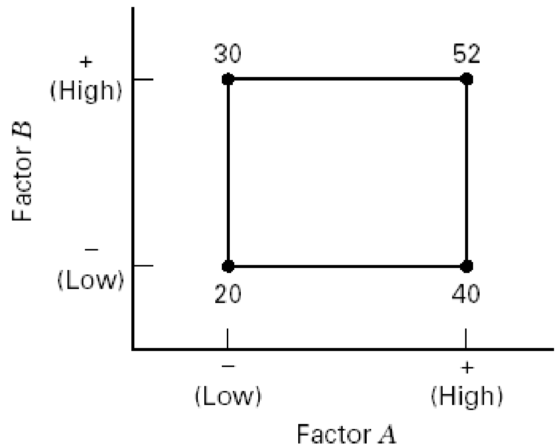


Figure 5-1 A two-factor factorial experiment, with the response (y) shown at the corners.

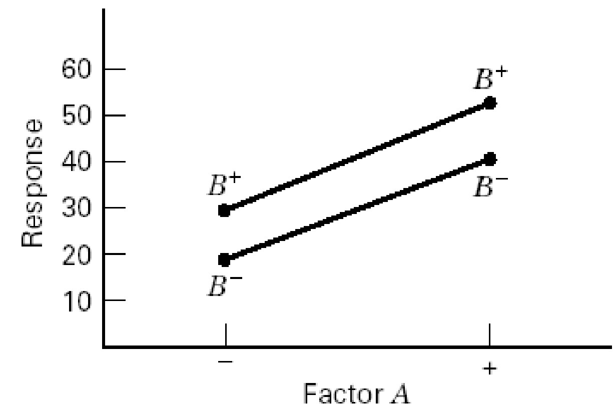


Figure 5-3 A factorial experiment without interaction.

Definition of a factor effect: The change in the mean response when the factor is changed from low to high

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{40+52}{2} - \frac{20+30}{2} = 21$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{30+52}{2} - \frac{20+40}{2} = 11$$

$$AB = \frac{52+20}{2} - \frac{30+40}{2} = -1$$

The Case of Interaction:

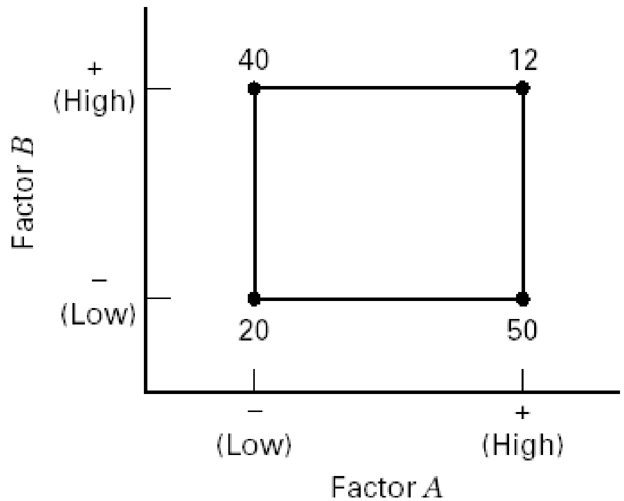


Figure 5-2 A two-factor factorial experiment with interaction.

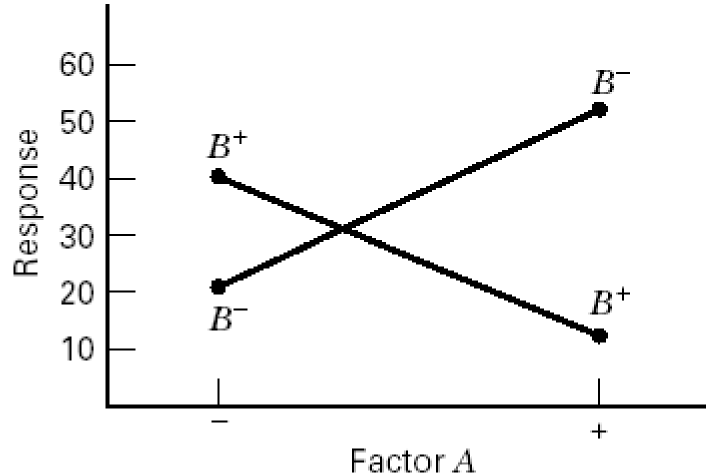


Figure 5-4 A factorial experiment with interaction.

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{50+12}{2} - \frac{20+40}{2} = 1$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{40+12}{2} - \frac{20+50}{2} = -9$$

$$AB = \frac{12+20}{2} - \frac{40+50}{2} = -29$$

Regression Model & The Associated Response Surface

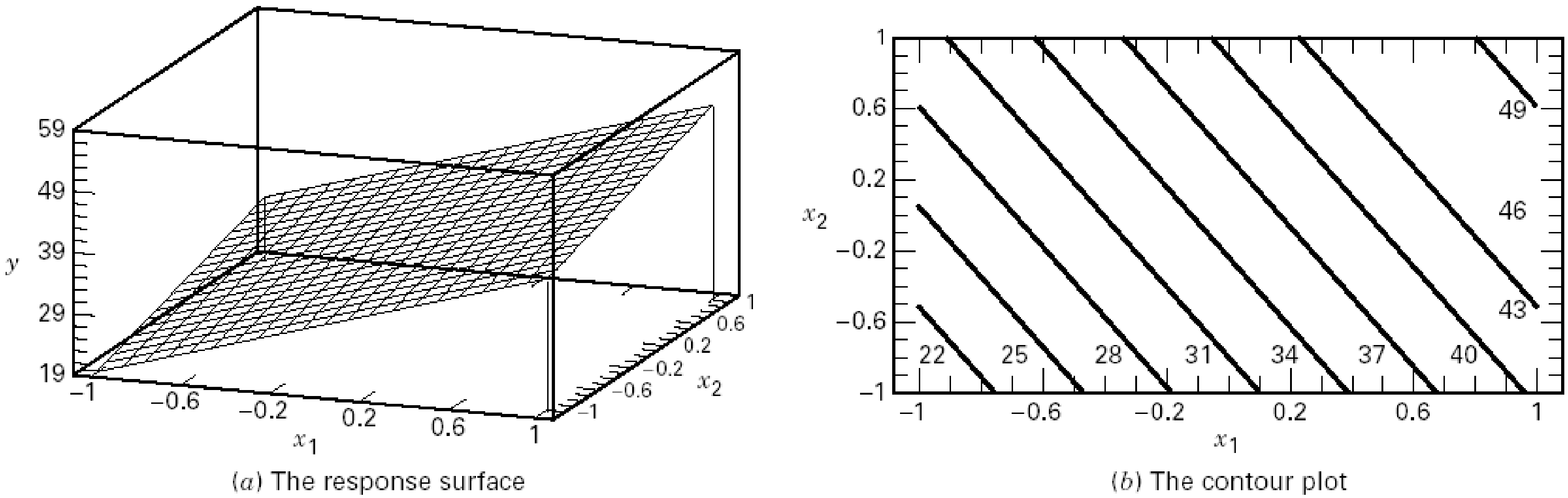


Figure 5-5 Response surface and contour plot for the model $\hat{y} = 35.5 + 10.5x_1 + 5.5x_2$.

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \varepsilon$$

The least squares fit is

$$\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 0.5x_1 x_2 \cong 35.5 + 10.5x_1 + 5.5x_2$$

The Effect of Interaction on the Response Surface

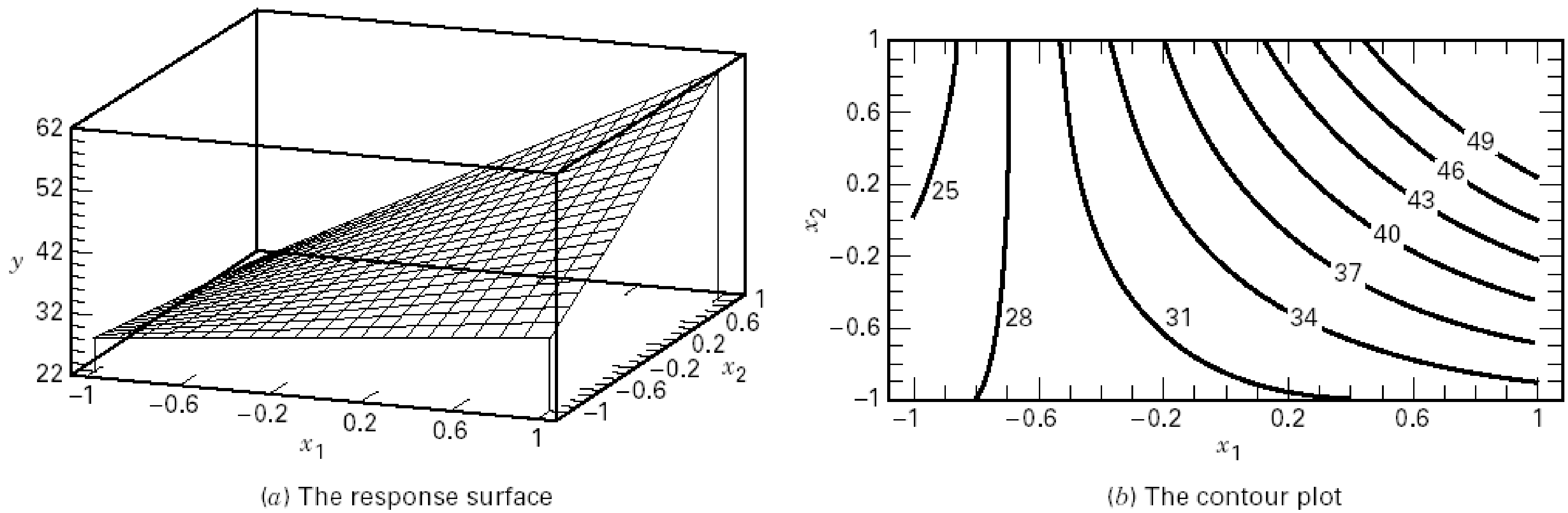


Figure 5-6 Response surface and contour plot for the model $\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 8x_1x_2$.

Suppose that we add an interaction term to the model:

$$\hat{y} = 35.5 + 10.5x_1 + 5.5x_2 + 8x_1x_2$$

Interaction is actually a form of **curvature**

Example 5-1 The Battery Life Experiment Text reference pg. 165

Table 5-1 Life (in hours) Data for the Battery Design Example

Material Type	Temperature (°F)					
	15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

A = Material type; *B* = Temperature (A **quantitative** variable)

1. What **effects** do material type & temperature have on life?
2. Is there a choice of material that would give long life ***regardless of temperature*** (a **robust** product)?

The General Two-Factor Factorial Experiment

Table 5-2 General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	...	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

a levels of factor *A*; *b* levels of factor *B*; *n* replicates

This is a **completely randomized design**

Table 5-2 General Arrangement for a Two-Factor Factorial Design

		Factor B			
		1	2	...	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

Statistical (effects) model:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

Other models (means model, regression models) can be useful

Extension of the ANOVA to Factorials (Fixed Effects Case) – pg. 177

$$\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\overline{y}_{...})^2 = bn\sum_{i=1}^{a}(\overline{y}_{i..}-\overline{y}_{...})^2 + an\sum_{j=1}^{b}(\overline{y}_{.j.}-\overline{y}_{...})^2$$

$$+n\sum_{i=1}^{a}\sum_{j=1}^{b}(\overline{y}_{ij.}-\overline{y}_{i..}-\overline{y}_{.j.}+\overline{y}_{...})^2 + \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk}-\overline{y}_{ij.})^2$$

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E$$

df breakdown:

$$abn-1 = a-1+b-1+(a-1)(b-1)+ab(n-1)$$

ANOVA Table – Fixed Effects Case

Table 5-3 The Analysis of Variance Table for the Two-Factor Factorial, Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a-1$	$MS_A = \frac{SS_A}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B treatments	SS_B	$b-1$	$MS_B = \frac{SS_B}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	$MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	$MS_E = \frac{SS_E}{ab(n-1)}$	
Total	SS_T	$abn-1$		

Design-Expert will perform the computations

Text gives details of **manual computing** (ugh!) – see pp. 169 & 170

Design-Expert Output – Example 5-1

Response: Life in hours
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	*F* Value	Prob > *F*	
Model	59416.22	8	7427.03	11.00	<0.0001	significant
A	*10683.72*	*2*	*5341.86*	*7.91*	*0.0020*	
B	*39118.72*	*2*	*19559.36*	*28.97*	*<0.0001*	
AB	*9613.78*	*4*	*2403.44*	*3.56*	*0.0186*	
Residual	18230.75	27	675.21			
Lack of Fit	*0.000*	*0*				
Pure Error	*18230.75*	*27*	*675.21*			
Cor Total	77646.97	35				

Std. Dev.	25.98	R-Squared	0.7652
Mean	105.53	Adj R-Squared	0.6956
C.V.	24.62	Pred R-Squared	0.5826
PRESS	32410.22	Adeq Precision	8.178

Residual Analysis – Example 5-1

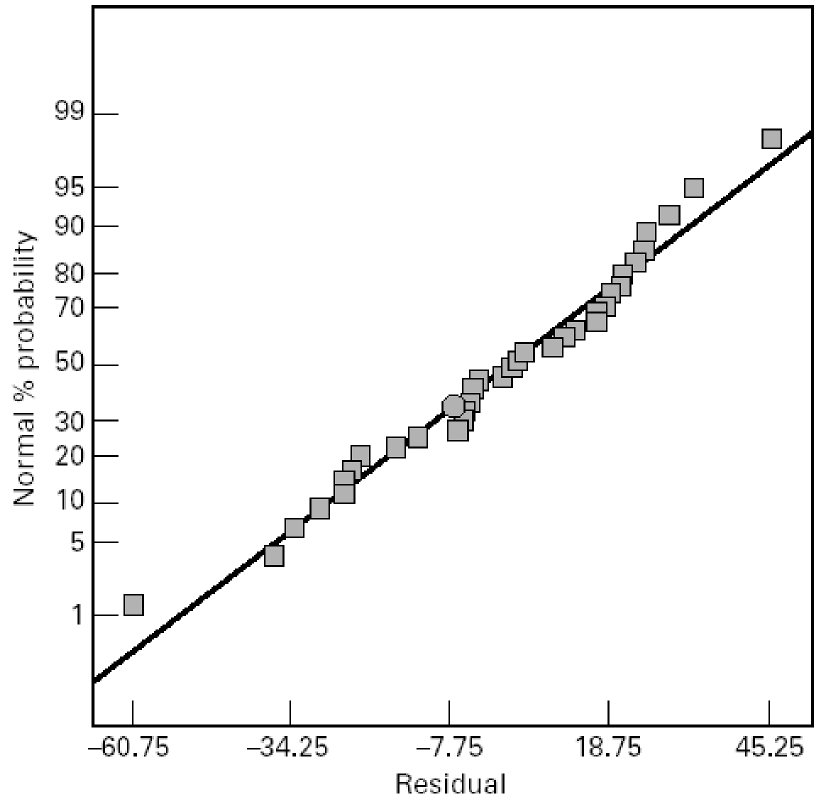


Figure 5-11 Normal probability plot of residuals for Example 5-1.

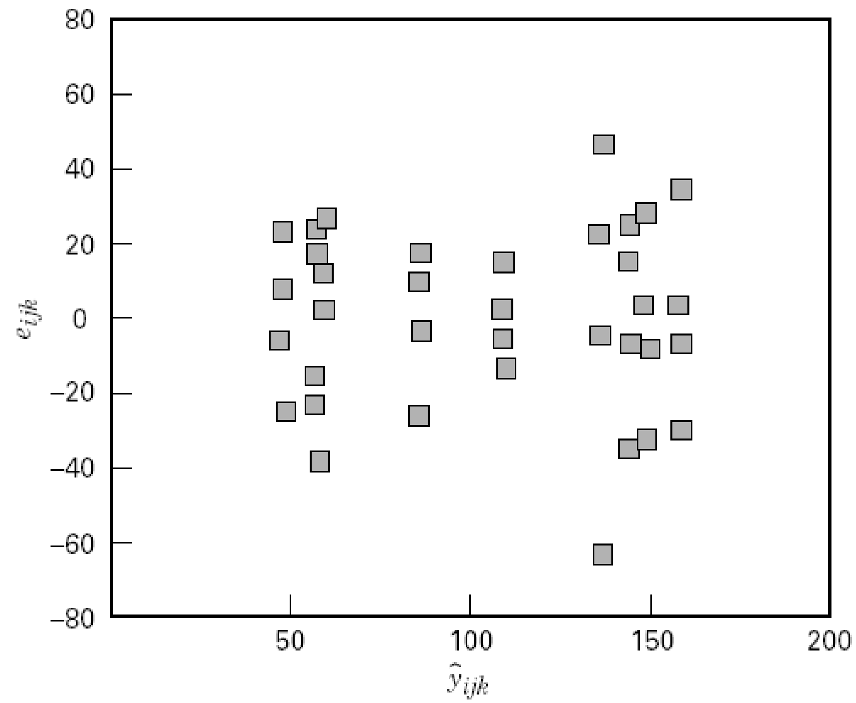


Figure 5-12 Plot of residuals versus $\hat{y}_{ijk}$ for Example 5-1.

Residual Analysis – Example 5-1

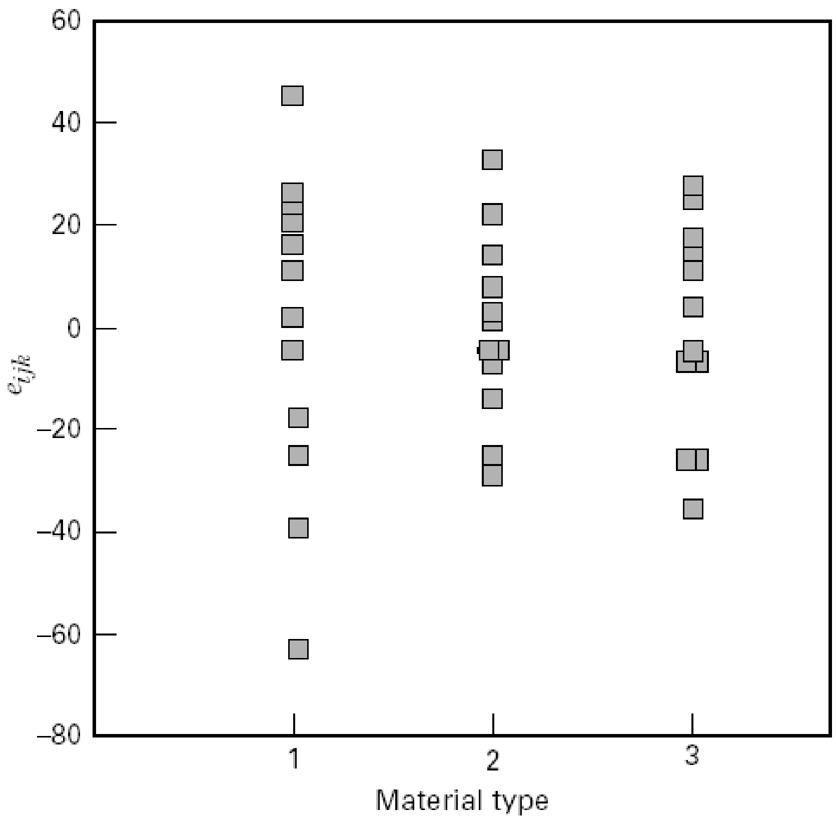


Figure 5-13 Plot of residuals versus material type for Example 5-1.

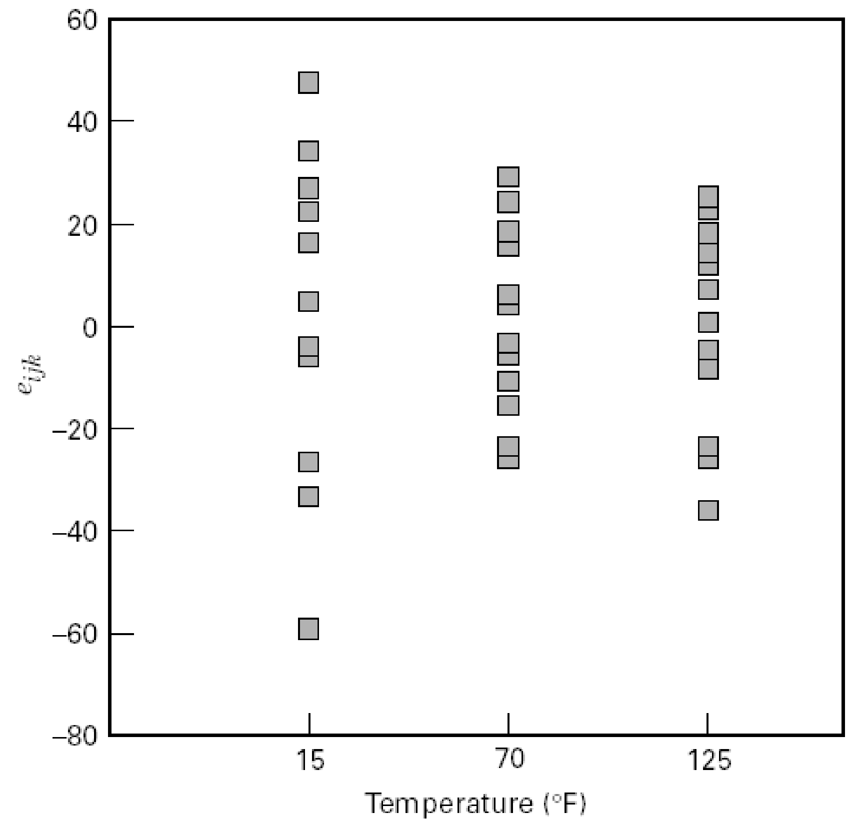
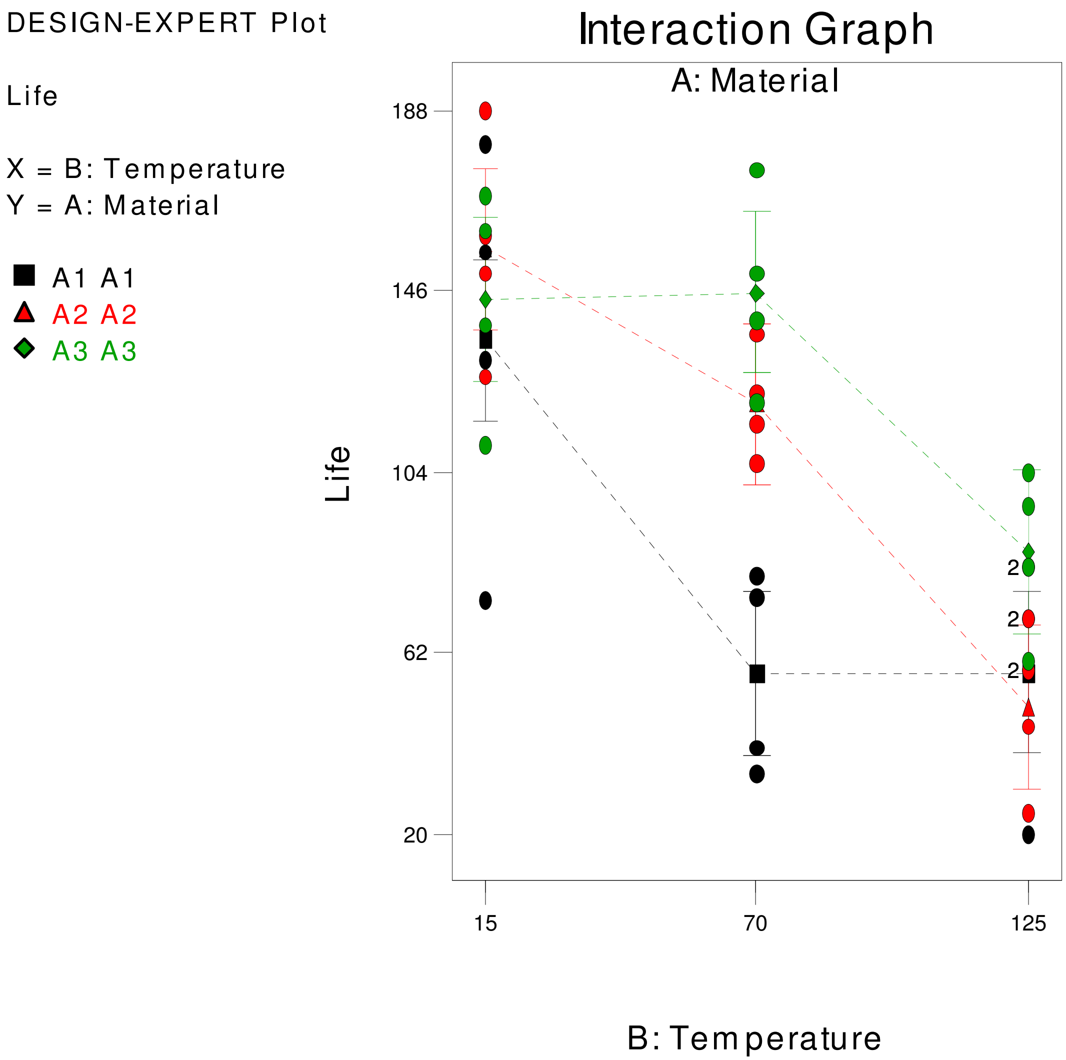


Figure 5-14 Plot of residuals versus temperature for Example 5-1.

Interaction Plot



Quantitative and Qualitative Factors

- The basic ANOVA procedure treats every factor as if it were **qualitative**
- Sometimes an experiment will involve both **quantitative** and **qualitative** factors, such as in Example 5-1
- This can be accounted for in the analysis to produce **regression models** for the quantitative factors at each level (or combination of levels) of the qualitative factors
- These **response curves** and/or **response surfaces** are often a considerable aid in practical interpretation of the results

Quantitative and Qualitative Factors

Table 5-15 Design-Expert Output for Example 5-4

Response: Life in hr
ANOVA for Response Surface Reduced Cubic Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	F Value	Prov > F	
Model	59416.22	8	7427.03	11.00	<0.0001	significant
A	*39042.67*	*1*	*39042.67*	*57.82*	<0.0001	
B	*10683.72*	*2*	*5341.86*	*7.91*	*0.0020*	
A^2	*76.06*	*1*	*76.06*	*0.11*	*0.7398*	
AB	*2315.08*	*2*	*1157.54*	*1.71*	*0.1991*	
A^2B	*7298.69*	*2*	*3649.35*	*5.40*	*0.0106*	
Residual	18230.75	27	675.21			
Lack of Fit	*0.000*	*0*				
Pure Error	*18230.75*	*27*	*675.21*			
Cor Total	77646.97	35				

Std. Dev.	25.98	R-Squared	0.7652
Mean	105.53	Adj R-Squared	0.6956
C.V.	24.62	Pred R-Squared	0.5826
PRESS	32410.22	Adeq Precision	8.178

Quantitative and Qualitative Factors

A = Material type

B = Linear effect of Temperature

B^2 = Quadratic effect of Temperature

AB = Material type – $\text{Temp}_{\text{Linear}}$

AB^2 = Material type - $\text{Temp}_{\text{Quad}}$

B^3 = Cubic effect of Temperature (Aliased)

Candidate model terms from Design-Expert:

Intercept

- A
- B
- B^2
- AB
- B^3
- AB^2

Regression Model Summary of Results

Final Equation in Terms of Actual Factors:

Material type 1

Life =

$+169.38017$
$-2.48860 * Temp$
$+0.012851 * Temp^2$

Material type 2

Life =

$+159.62397$
$-0.17901 * Temp$
$+0.41627 * Temp^2$

Material Type 3

Life =

$+132.76240$
$+0.89264 * Temp$
$-0.43218 * Temp^2$

Regression Model Summary of Results

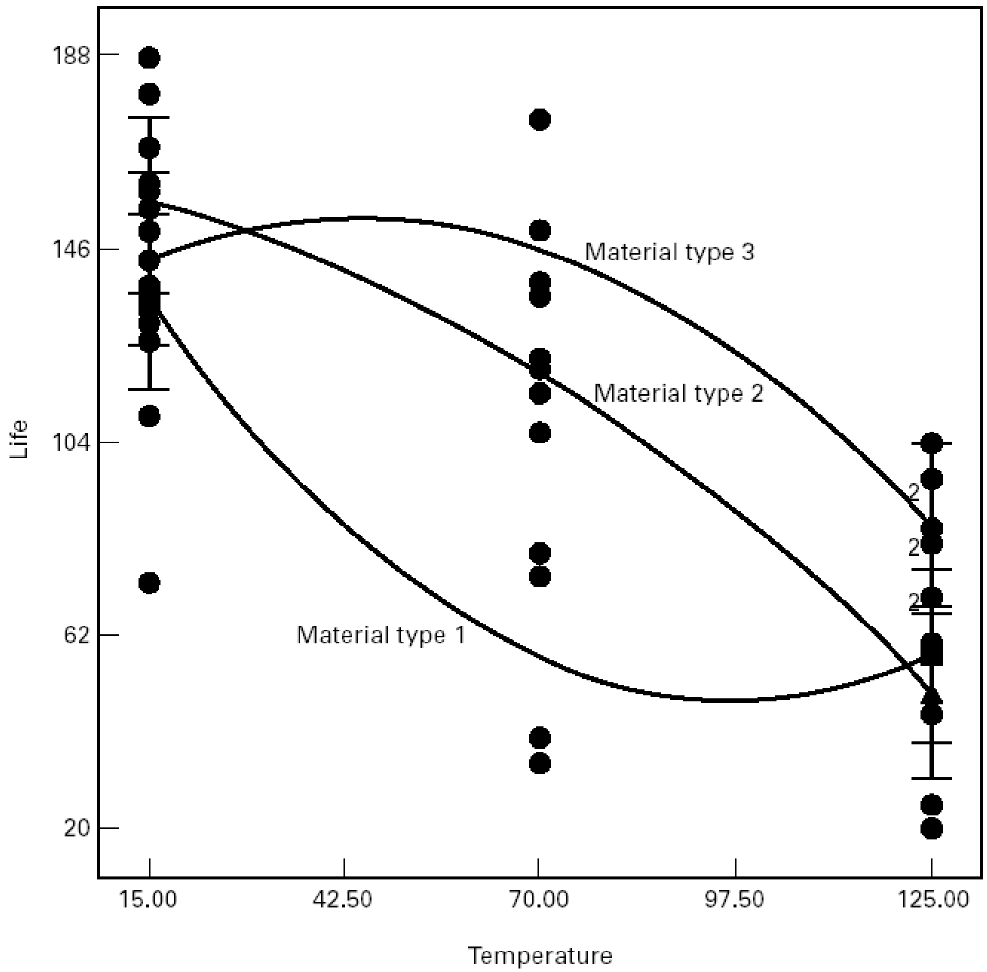


Figure 5-18 Predicted life as a function of temperature for the three material types, Example 5-4.

Factorials with More Than Two Factors

- Basic procedure is similar to the two-factor case; all *abc…kn* treatment combinations are run in random order
- ANOVA identity is also similar:

$$SS_T = SS_A + SS_B + \cdots + SS_{AB} + SS_{AC} + \cdots$$
$$+ SS_{ABC} + \cdots + SS_{AB\cdots K} + SS_E$$

- Complete three-factor example in text, Example 5-5